Using the Galerkin method to compute the eigenvalues and eigenelements of the second-order Sturm–Liouville problems

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A B S T R A C T

In this paper, an efficient method based on the Galerkin technique for computing the eigenvalues and eigenfunctions for the second-order Sturm-Liouville problems. The first kind of Chebyshev polynomials $T_n(x)$ is used as basis functions to solve this problem. The Chebyshev-Galerkin method is applied to reduce an ordinary differential equation into a system of algebraic equations using the orthogonality of Chebyshev polynomials and new relations driven from the orthogonality property. Numerical examples show that the proposed method is an easy method to implement and introduce accurate results.

1 Introduction

Sturm-Liouville boundary value problem has played significant roles in many areas of science, engineering, and mathematics. The researchers have studied and many of the associated theories introduced over 170 years ago. The solution of various problems in the fields of mathematics, physics, and engineering is closely related to the solution of Sturm-Liouville problem. In mathematics, some separable partial differential equations have reduced to Sturm-Liouville problem as wave and heat equations that are used for describing vibrations of beams of structures and heat transferred by conduction in rod in addition to many differential equations such as Bessel, Hermite, Jacobi, and Legendre equations can be transformed into Sturm-Liouville equations as in [1, 2]. In physics, the solution of Sturm-Liouville problems

provides a solution of Schrödinger's equation that is a key result in quantum mechanics and illustrates energy state and energy spectrum of systems, for example, hydrogen atoms. The Sturm-Liouville theory has been the basis for the development of fields such as the spectral theory of differential operators, functional space and modern analysis and theory of self-adjoint operators [3].

Many methods have been developed to solve one-dimensional Sturm-Liouville problems, among these methods, finite difference methods [4, 5]. S. Pruess replaced the coefficients of Sturm-Liouville equation with piecewise polynomials functions to solve it. Recently, Hargrave [7] solved linear Sturm-Liouville eigenvalue problems by reducing the second-order Sturm- Liouville into a system of first-order equations by using Prüfer substitutions. Canosa and Oliveira [8] presented a one-dimensional Schrödinger equation that represents the second-order Sturm-Liouville and solved it by using step function. Wittrick and Williams [9] developed an efficient algorithm for computing the natural undamped frequencies (eigenvalues) of vibration of any linearly elastic structure. Jiang et al. [10] studied the existence of second-order singular Sturm–Liouville integral boundary value problems. S. Abbasbandy and A. Shirzadi [11] used the homotopy analysis method for calculating the second and fourth-order Sturm-Liouville. S. Yuan et al. [12] used the exact dynamic stiffens method for solving regular second-fourth-order Sturm-Liouville problem. B. Chanane [13] used a method based on sampling theory to compute the eigenvalues of second-order Sturm-Liouville problems with parameter-dependent potential and boundary conditions and modified his method to be valid for fourth-order Sturm-Liouville as in [14]. M. I. Syam and H. I. Siyyam [15, 16] used variational iteration methods for computing the eigenvalues of fourth and sixth-order Sturm-Liouville problem. B. S. Attili and D. Lesnic[17] used the Adomian decomposition method for getting the eigenelments of fourth-order Sturm-Liouville problem. Based on finite difference and Numerov's methods A. Rattana and C. Böckmann [18] stated their method for computing fourth-order Sturm-Liouville. M. El-Gamel et al. [19] used Galerkin technique and Legendre polynomials as basis functions for computing the eigenelements of fourth-order Sturm-Liouville.

In this paper, we present a Chebyshev-Galerkin method for solving the second-order Sturm-Liouville boundary value problem in the form:

$$\frac{d}{dx}\big[p(x)y'(x)\big]+q(x)y(x)=\lambda w(x)y(x),\quad -1\le x\le 1, \tag{1.1}$$

subject to the boundary conditions

$$y(-1)=y(1)=0. \tag{1.2}$$

where $p(x)$, $q(x)$ and $w(x)$ are continuous functions. This equation contains both an unknown function called eigenfunctions $y(x)$ and an unknown parameter called eigenvalues λ. By using our method, we won't compute the eigenvalues of this problem only, but also we will find eigenfunction at different values λ.

This article is organized as follows: a brief introduction the preliminaries of Chebyshev polynomials is presented in Section 2. How to use the Chebyshev-Galerkin method to solve the Sturm-Liouville problem introduces in Section 3. The treatment of general solution domain for any problem is introduced in Section 4. Section 5 includes four solved examples of second-order Sturm–Liouville problems. Finally, the conclusions of our results.

2 Chebyshev polynomials Preliminaries

In this section, we present some necessary definitions and mathematical preliminaries of Chebyshev polynomials first kind $T_n(x)$ and some important properties that are required for our subsequent development. The well-known Chebyshev polynomials are defined on the interval $[-1, 1]$ and can be determined at endpoints:

$$T_n(-1) = (-1)^n, \quad T_n(1) = 1. \tag{2.1}$$

The power function x^k can be expanded in a Chebyshev series form as:

$$x^k = 2^{1-k} \sum_{\substack{j=0 \\ k-j \text{ even}}}^{k}{}' \frac{k!}{\left(\dfrac{k-j}{2}\right)!\left(\dfrac{k+j}{2}\right)!} T_j(x). \tag{2.2}$$

The product between any two Chebyshev polynomials is given by:

$$2T_n(x)T_m(x) = T_{n+m}(x) + T_{|n-m|}(x). \tag{2.3}$$

The orthogonality of polynomials on $[-1, 1]$ is given by [20-22]:

$$\int_{-1}^{1} T_n(x)T_m(x) \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2} c_n \delta_{n,m}, \forall n, \, m \geq 0, \tag{2.4}$$

where $\delta_{n,m}$ is the Kronecker symbol and

$$c_n = \begin{cases} 2, & if \ n=0, \\ 1, & if \ n \geq 1, \\ 0, & otherwise. \end{cases}$$

The sth derivative of Chebyshev polynomials $T_n(x)$ is given as [23] :

$$T_n^{(s)}(x) = 2^s \sum_{j=0}^{\frac{n-s}{2}} n(n-j-1)\underline{s-1} \ \frac{(s+j-1)!}{j!(s-1)!} T_{n-2j-s}(x)$$

$$-2^{s-1}n[n-s \ even]\left(n+\frac{s}{2}-1\right)\underline{s-1} \ \frac{\left(n+\frac{s}{2}-1\right)!}{(s-1)!\left(n+\frac{s}{2}\right)!}T_0(x), \tag{2.5}$$

where falling factorials $(x)\underline{n} = x(x-1)\cdots(x-n+1)$ and inversion symbol $[p]$ which is true 1 if p and 0 otherwise.

The first $T_n'(x)$ and second derivative $T_n''(x)$ of Chebyshev polynomial are reduced from above equation.

$$T_n' = 2\sum_{j=0}^{\frac{n-1}{2}} nT_{n-2j-1}(x) - n[n-1 \ even] \ T_0(x). \tag{2.6}$$

$$T_n'' = 4\sum_{j=0}^{\frac{n-2}{2}} n(n-j-1)(j+1)T_{n-2j-2}(x) - 4[n-2 \ even]\left(\frac{n}{2}\right)^3 T_0(x). \tag{2.7}$$

The sth derivative of the polynomials at endpoints is given as [24]:

$$\left.\frac{d^s T_n(x)}{dx^s}\right|_{x=\pm1} = (\pm1)^{n+s}\prod_{k=0}^{s-1}\frac{n^2-k^2}{2k+1}, n \geq s. \tag{2.8}$$

3 Chebyshev-Galerkin method

3.1 Eigenvalues computation

To compute the eigenvalues of equation (1.1) with boundary conditions (1.2) by using Chebyshev polynomials and Galerkin technique together, we will introduce the

integration value of the product between x^r, $T_n(x)$, and $T_n^{(s)}(x)$ using the Chebyshev orthogonality.

Lemma 1 *For n, m and r are integer values*

(i) $\displaystyle\int_{-1}^{1} \frac{x^r T_n(x) T_m(x)}{\sqrt{1-x^2}} dx$

$$= \frac{1}{2^r} \sum_{\substack{p=0 \\ r-p \text{ even}}}^{r}{}' \frac{r!}{\left(\dfrac{r-p}{2}\right)!\left(\dfrac{r+p}{2}\right)!} \frac{\pi}{2} c_p (\delta_{p,m+n} + \delta_{p,|m-n|}).$$

(ii) $\displaystyle\int_{-1}^{1} \frac{x^r T_n'(x) T_m(x)}{\sqrt{1-x^2}} dx$

$$= 2^{1-r} \sum_{j=0}^{\frac{n-1}{2}} \sum_{\substack{p=0 \\ r-p \text{ even}}}^{r}{}' n \frac{r!}{\left(\dfrac{r-p}{2}\right)!\left(\dfrac{r+p}{2}\right)!} \frac{\pi}{2} c_p (\delta_{p,n-2j+m-1} + \delta_{p,|n-2j-m-1|})$$

$$- 2^{1-r}[n-1 \text{ even}] n \sum_{\substack{p=0 \\ r-p \text{ even}}}^{r}{}' \frac{r!}{\left(\dfrac{r-p}{2}\right)!\left(\dfrac{r+p}{2}\right)!} \frac{\pi}{2} c_p \delta_{p,m}.$$

(iii) $\displaystyle\int_{-1}^{1} \frac{x^r T_n''(x) T_m(x)}{\sqrt{1-x^2}} dx$

$$= \frac{2^{2-r} \pi}{2} \sum_{j=0}^{\frac{n-2}{2}} \sum_{\substack{p=0 \\ r-p \text{ even}}}^{r}{}' \frac{n(n-j-1)(j+1)r! \, c_p}{\left(\dfrac{r-p}{2}\right)!\left(\dfrac{r+p}{2}\right)!} \left(\delta_{p,n-2j+m-2} + \delta_{p,|n-2j-m-2|}\right)$$

$$- \frac{2^{3-r} \pi}{2}[n-2 \text{ even}] \left(\frac{n}{2}\right)^3 \sum_{\substack{p=0 \\ r-p \text{ even}}}^{r}{}' \frac{r!}{\left(\dfrac{r-p}{2}\right)!\left(\dfrac{r+p}{2}\right)!} c_p \delta_{p,m}.$$

Proof

(i) By recalling equations (2.2), (2.3) and (2.4), we can write x^r in terms of $T_j(x)$. By using equation (2.3), the product of $T_n(x) T_m(x)$ can be as a sum of two Chebyshev polynomials. Finally, by using the orthogonality property (2.4), we can prove (*i*). To prove lemma (ii) and (iii), $T_n'(x)$ and $T_n''(x)$ can be written as $\sum T(x)$ using equation

(2.6) and (2.7) respectively. Similar to the steps in lemma (i), relations (ii) and (iii) can be proved.

For the sake of simplicity, equation (1.1) can be expanded and written as:

$$p(x)y''(x) + p'(x)y'(x) + q(x)y(x) = \lambda w(x)y(x), \quad -1 \le x \le 1. \tag{3.1}$$

The approximated solution of equation (1.1) in terms of the first kind Chebyshev polynomials $T_n(x)$ is given as:

$$y_n(x) \cong \sum_{j=0}^{n} c_j T_j(x). \tag{3.2}$$

By applying Galerkin method using Chebyshev polynomials as a basis on equation (3.1), yields

$$\langle p(x)y''(x), T_r(x) \rangle + \langle p'(x)y'(x), T_r(x) \rangle + \langle q(x)y(x), T_r(x) \rangle = \lambda \langle w(x)y(x), T_r(x) \rangle, \tag{3.3}$$

where the inner product $\langle ., . \rangle$ is defined as:

$$\langle \varphi(x), \psi(x) \rangle = \int_{-1}^{1} \varphi(x)\psi(x) \frac{dx}{\sqrt{1-x^2}}.$$

By replacing each $y(x)$ in the equation (3.3) with the approximate solution defined in (3.2) and expanding the functions $p(x)$, $p'(x)$, $q(x)$ and $w(x)$, we introduce the following theorem.

Theorem 3.1 *If the assumed approximate solution of (1.1) with boundary condition (1.2) is (3.2), then the discrete Chebyshev-Galerkin system for the determination of the unknown coefficients $\{c_j\}_{j=0}^{n}$ is given by:*

$$\sum_{j=0}^{n} \left[\sum_{\kappa_1} a_{\kappa_1} \int_{-1}^{1} \frac{x^{\kappa_1} T_j''(x) T_r(x)}{\sqrt{1-x^2}} dx + \sum_{\kappa_2} a_{\kappa_2} \int_{-1}^{1} \frac{x^{\kappa_2} T_j'(x) T_r(x)}{\sqrt{1-x^2}} dx \right.$$

$$\left. + \sum_{\kappa_3=0} a_{\kappa_3} \int_{-1}^{1} \frac{x^{\kappa_3} T_j(x) T_r(x)}{\sqrt{1-x^2}} dx \right] c_j = \lambda \sum_{j=0}^{n} \sum_{\kappa_4=0} \left[a_{\kappa_4} \int_{-1}^{1} \frac{x^{\kappa_4} T_j(x) T_r(x)}{\sqrt{1-x^2}} dx \right] c_j.$$

The matrix form of the previous theorem is

$$A\mathbf{c} = \lambda B\mathbf{c}, \tag{3.4}$$

where

$$A = \begin{pmatrix} S \\ E \end{pmatrix}, \quad B = \begin{pmatrix} G \\ O \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix},$$

$$\{S\}_{j,r} = \sum_{\kappa_1}^{q} a_{\kappa_1} \int_{-1}^{1} \frac{x^{\kappa_1} T_j''(x) T_r(x)}{\sqrt{1-x^2}} dx$$

$$+ \sum_{\kappa_2}^{q} a_{\kappa_2} \int_{-1}^{1} \frac{x^{\kappa_2} T_j'(x) T_r(x)}{\sqrt{1-x^2}} dx + \sum_{\kappa_3}^{q} a_{\kappa_3} \int_{-1}^{1} \frac{x^{\kappa_3} T_j(x) T_r(x)}{\sqrt{1-x^2}} dx,$$

$$\{G\}_{j,r} = \sum_{\kappa_4}^{q} a_{\kappa_4} \int_{-1}^{1} \frac{x^{\kappa_4} T_j(x) T_r(x)}{\sqrt{1-x^2}} dx.$$

and $\{E\}_j = \sum_{j=0}^{n} c_j T(\pm 1)$ which is deduced from the boundary conditions (1.2) and can be determined from equation (2.1). The terms $\{S\}_{j,r}$ and $\{G\}_{j,r}$ are calculated by using equations Lemma 1. By multiplying equation (3.4) by B^{-1}, yields

$$\phi \mathbf{c} = \lambda \mathbf{c}, \qquad (3.5)$$

where $\phi = B^{-1}A$. Equation (3.5) be in homogenous form as

$$(\phi - \lambda I)\mathbf{c} = O. \qquad (3.6)$$

So, the eigenvalues λ can be calculated by solving the equation

$$|\phi - \lambda I| = 0.$$

3.2 Eigenfunctions computation

This section illustrates how eigenfunctions can be calculated. The basic idea of calculations is converting the equation (1.1) into an initial value problem. Assume that $y_1(x)$ and $y_2(x)$ are the solutions of initial value problem of equation (1.1) by normalizing conditions.

$$y(-1) = 0, \quad y'(-1) = 1,$$

and

$$y(-1) = 0, \quad y'(-1) = 0,$$

respectively. The eigenfunctions can be obtained:

$$y_n(x) = y_1(x) + \beta\, y_2(x), \qquad (3.7)$$

According to equation (1.2), equation (3.11) can be reformed as:

$$y_n(x) = y_1(x) - \frac{y_1(1)}{y_2(1)} y_2(x), \qquad \beta = -\frac{y_1(1)}{y_2(1)}.$$

4 Treatment of General solution domain

The solution domain changes from application to another. In several applications, the solution domain is not necessary to be $[-1, 1]$. To solve Sturm-Liouville problem using Chebyshev-Galerkin method discussed in Section 3 with a general solution domain $[a,b]$, we should use the linear transformation:

$$t = \frac{1}{2}(b-a)(x+1) + a. \qquad (4.1)$$

Assume, the general domain of the second-order Sturm-Liouville problem as:

$$p(t)y''(t) + p'(t)y'(t) + q(t)y(t) = \lambda w(t)y(t), \quad a \le t \le b. \qquad (4.2)$$

By applying the transformation (4.1) into the problem (4.2), yields

$$\left(\frac{2}{b-a}\right)^2 p(x)y''(x) + \left(\frac{2}{b-a}\right)p'(x)y'(x) + q(x)y(x) = \lambda w(x)y(x), \quad -1 \le x \le 1.$$

After solving the last equation by our method, we can get the solution in terms of variable t by using the inverse linear transformation that is used above, we can get the eigenfunctions in terms of x.

5 Numerical results

In this section, four examples of second-order Sturm-Liouville problems are presented to confirm the accuracy of the proposed method. The eigenvalues and corresponding eigenfunctions for each problem are calculated. The comparison between the results of Chebyshev-Galerkin method and other methods will introduce in this section. All programs carry out by using MATLAB 2012 and Mathematica 10 packages.

Example 5.1 [25, 26] Consider the following second-order Sturm-Liouville problem

$$y''(x) + \lambda y(x) = 0, \quad 0 \le x \le 1, \tag{5.1}$$

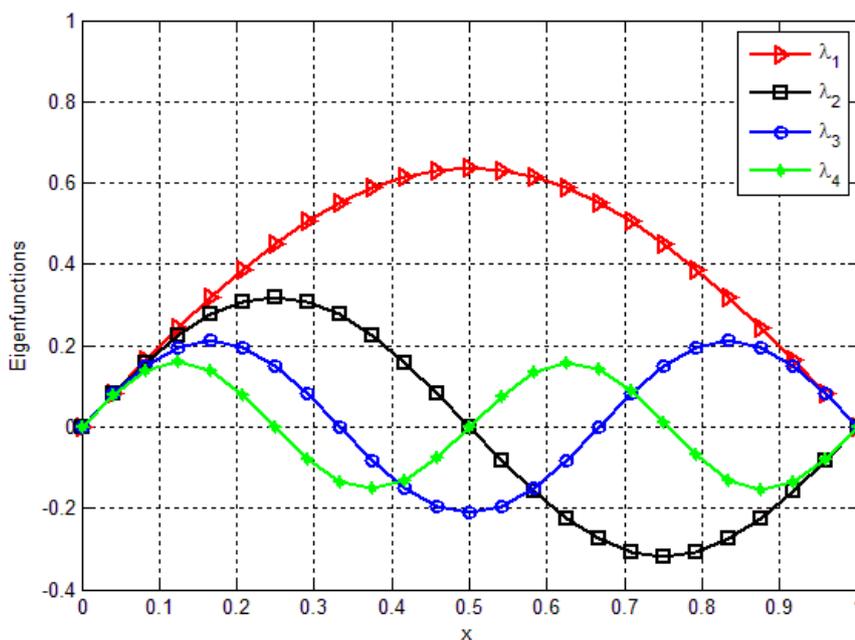with boundary conditions

$$y(0) = 0, \quad y(1) = 0.$$

The exact solution is given in [25, 26]

$$\lambda = (k\pi)^2, \quad k = 1, 2, 3, \ldots$$

The eigenvalues of this problem are displayed in Table 1 beside the exact solution. Fig. 1 shows the eigenfunctions for the first four eigenvalues

Table 1: Comparison of eigenvalues for example 5.1

k	$\lambda_{Chebyshev-Galerkin}$	λ_{Exact}
1	9.869604401089358	9.869604401089358
2	39.47841760435743	39.47841760435743
3	88.82643960980422	88.82643960980422
4	157.9136704174297	157.9136704174297
5	246.7401100272339	246.7401100272339
6	355.3057584392169	355.3057584392169

Fig. 1: The first four eigenfunctions for Example 5.1

Example 5.2 [5, 6] Consider the following second-order Sturm-Liouville problem

$$x^6 y''(x) + 0.75 x^4 y(x) + \lambda y(x) = 0, \quad 1 \le x \le 2, \tag{5.2}$$

with boundary conditions

$$y(1) = 0, \quad y(2) = 0.$$

The exact solution is given in [5, 6] as $\lambda = \dfrac{64}{9} k^2 \pi^2$, $k = 1,\ 2,\ 3,\ \dots..$ The comparison between the eigenvalues of equation (5.2) and exact values are listed in Table 2. The eigenfunctions of the first four eigenvalues are presented in Fig. 2.

Table 2: Comparison of eigenvalues for example 5.2

k	$\lambda_{Chebyshev\text{–}Galerkin}$	λ_{Exact}
1	65.646421082678754	70.183853518857661
2	275.34349367762262	280.73541407543064
3	625.97865842322415	631.65468166971895
4	1117.1401432999673	1122.9416563017225
5	1748.7294360422866	1754.5963379714415
6	2520.7137863704301	2526.6187266788758
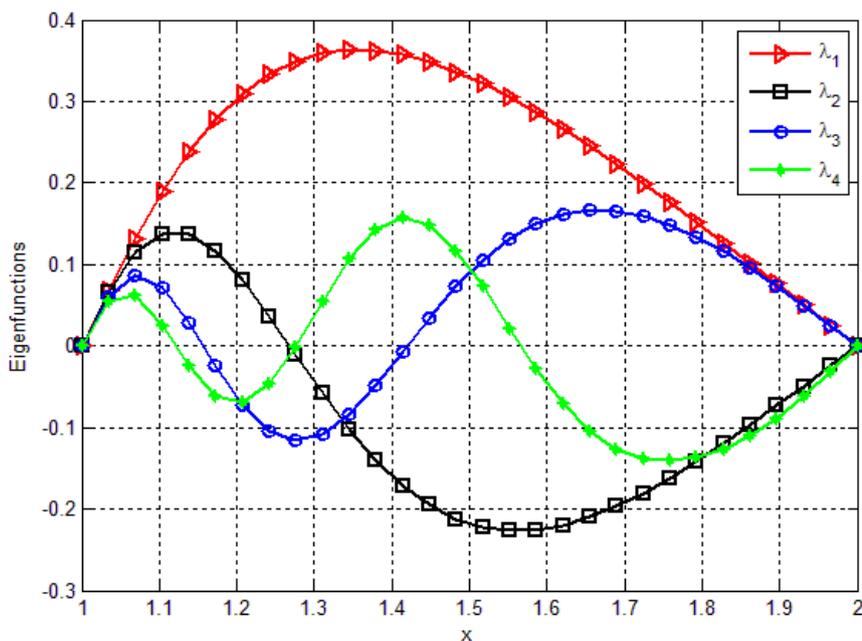
Fig. 2: The first four eigenfunctions for Example 5.2

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Example 5.3 [17] Consider the following second-order Sturm-Liouville problem

$$y''(x) - 0.01x^2 y(x) + \lambda y(x) = 0, \quad 0 \le x \le 5, \tag{5.3}$$

subject to

$$y(0) = 0, \quad y(5) = 0$$

Equation (5.3) plays a great role in solving the eigenvalues of higher order fourth and sixth order Sturm-Liouville as mention in [27], [17] and [24]. The eigenvalues of fourth-order Sturm-Liouville can be determined by square of the eigenvalues of equation (5.3) and the eigenvalues of sixth-order Sturm-Liouville can be computed by cubic the eigenvalues of equation (5.3). In Table 3, the computed eigenvalues are compared with those obtained by Adomian decomposition method and SLEIGN2 algorithm [29]. Fig. 3 shows the first four eigenfunctions.

Example 5.4 [11] Consider the following second-order Sturm-Liouville problem

$$xy''(x) + y(x) + \lambda xy(x) = 0, \quad 0 < x < 1, \tag{5.4}$$

with boundary conditions

$$y(0) = 0, \quad y(1) = 0.$$

S. Abbasbandy and A. Shirzadi [11] used the homotopy method to solve this problem, while P. Baily et al. [29] used SLEIGN2 algorithm to compute the eigenvalues of above problem. Table 4 shows a comparison between the eigenvalues of equation (5.4) that are computed by our method and the previous methods. Fig. 4 presents the first four eigenfunction.

**Table 3: Comparison of eigenvalues for example 5.3**

k	$\lambda_{Chebyshev-Galerkin}$	λ_{ADM}	$\lambda_{SLEIGN2}$
1	0.46373576996809	0.46373576999161	0. 463735819000
2	1.65976201145158	1.65976201145916	1. 659762140000
3	3.63529249725374	3.63529249724954	
4	6.39928275351277	6.39928275349180	
5	9.95256138202374	9.95256138580108	
6	14.2953045531725	14.2952612249506	
7	19.4275685737352	19.7523934066243	
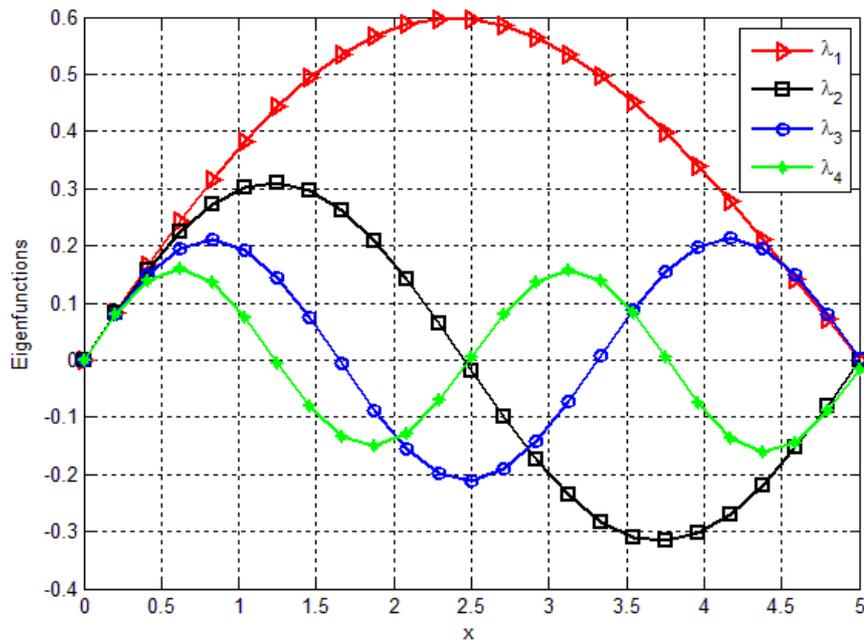
Fig. 3: The first four eigenfunctions for Example 5.3

Table 4: Comparison of eigenvalues for example 5.4

k	$\lambda_{Chebyshev–Galerkin}$	$\lambda_{Homotopy}$	$\lambda_{SLEIGN\ 2}$
1	7.3739850191	7.3739850100	7.3739900000
2	36.336019594	36.336018510	36.336020000
3	85.292582096	85.292510750	85.292580000
4	154.09862374	154.10192997	154.09862000
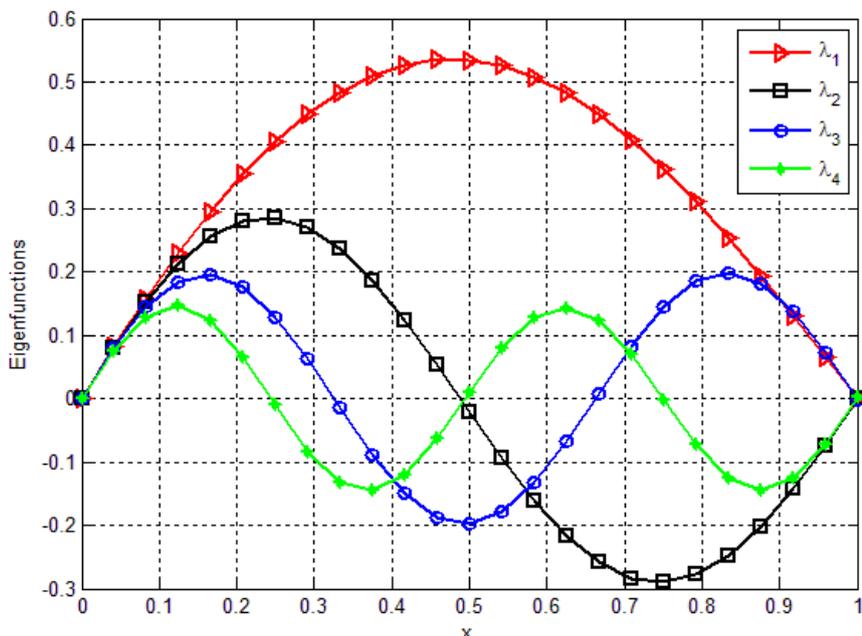
Fig. 4: The first four eigenfunctions for Example 5.4

6 Conclusion

In this paper, we introduced an accurate Chebyshev-Galerkin numerical technique for evaluating the eigenvalues and eigenfunctions of second-order Sturm-Liouville boundary value problems. The numerical examples illustrate the efficiency and accuracy of the present method as follows: in Example 1, the eigenvalues are very close to the exact solution. The results are in good agreement which is approaching to the exact solution as in Example 2. The results of Examples 3, 4 are in good agreement with the other methods like Adomian decomposition, homotopy and SLEIGN2 algorithm.

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