On The Studying of Contact Mechanics of a Flat Stamp Graded Coatings Using An Advanced Barycentric Lagrange Interpolation Formula

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Abstract

In this paper, the initial fracture in graded coatings under sliding rigid flat stamp contact loading is considered. This problem could be represented in the form of weakly singular Fredholm of the second kind. Shoukralla et al [1], made progress towards the application of some advanced single and double barycentric Lagrange interpolants and how to adapt them to be applicable to completely isolating the kernel singularity and find accurate solutions of the weakly singular Fredholm equations of the second kind which we will use in this paper for studying the effect of many factors such as the coefficient of friction, material inhomogeneity constants, and other length parameters on the critical stresses that may affect the break of the coating.

1. Introduction

The functionally graded materials (FGMs) possess thermo-mechanical properties and continuously varying volume fractions such that, it appears in many applications Such as bearings, micro-electromechanical systems, gears, transportation, radiation, automobile, magnetic disk drives, energy harvesting, aerospace, cams, and abradable seals in gas turbines [2], [3], [4]. As a result of that, many scientists began to be interested in studying (FGMs). İsa Çömez. [5] used linear elasticity theory to clarify the contact problem of a functionally graded layer loaded by a rigid cylindrical punch and supported by a Winkler foundation. In 2015, [6] he made a great revolution and was Firstly discussed a new problem by considering that the punch is affected by a concentrated normal force and moves with a subsonic constant velocity. The calculated equations are reduced to a Cauchy singular integral equation using Fourier transform and solved using the formulas of Gauss–Chebyshev integration. George G. Adams. [7] declared and studied the change in the analysis of contact problem If a rigid punch is perfectly bonded to an elastic half-plane where the stress state gives a well-known oscillating singularity. A. V. Loveikin. [8] developed a great analysis using Fourier transformation and expansion–collocation technique for the analysis of the contact mechanics of functionally graded materials (FGMs) which have elastic gradation in the sided direction. Scientists discovered that many of (FGMs) application problems can be reduced to a specific type of equations. M. A. Guler *et al*. In 1980, I. Bakirtas. [9] investigated the problem of a rigid punch on an elastic half space and used Fourier Transform Technique to reduce it to a singular integral equation. I. Comez · M. A. Guler.[10], considered the functionally graded bilayer in investigating the plane contact problem for a rigid cylindrical punch. Fourier integral transform is used to reduce that problem to a singular integral equation which is solved numerically by Gauss–Jacobi formula to find the unknown contact width and the contact pressure. [11], investigated the effect of contact stress obtained from graded half-plane sliding frictional contact mechanic problem. For the circular stamp profile, the stress intensity and contact stresses factors are numerically calculated numerically in the form of a singular Fredholm integral equation of the second type. We will investigate in this paper the initial fracture in graded coatings under sliding rigid flat stamp contact loading using the Barycentric Lagrange Interpolation technique [1].

2. Formulation of the problem

Consider the FGM-coated elastic half-plane with an ordinary stress boundary value problem described in Fig. (1). The structural is a metallic substrate bonded to a metal/ceramic coating with continuous and variable thermo-mechanical properties. Let $\mu(y)$ is the shear modulus of the coating given by

$$\mu(y)=\mu_1 e^{\gamma y}, \qquad -h<y<0, \tag{1}$$

where h is the thickness of the graded coating (medium 1), γ is a constant related to the inhomogeneity of the material given by Eq.(3), medium 2 is a homogeneous substrate, μ_2 is a constant shear modulus of the substrate of the form

$$\mu_2=\mu_1 e^{-\gamma y}, \qquad -h<y<0, \tag{2}$$

$$\gamma=\frac{1}{h}\log\Gamma_3,\ \Gamma_3=\frac{\mu_2}{\mu_1}, \tag{3}$$

where μ_2 equals to $\mu(y)$ at the surface. Assume that the Poisson's ratio is negligible. As a result of that $v_2=v_1(y)=v=$constant.

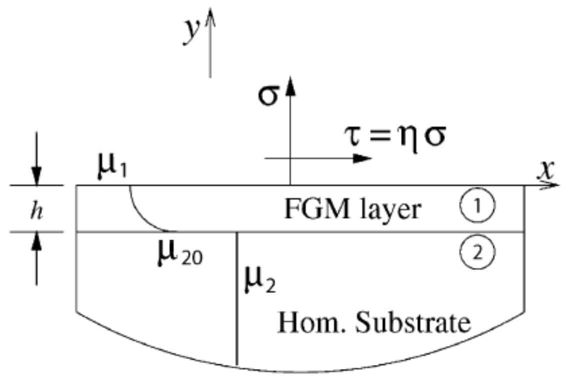

Fig.(1) Geometry of the problem for an FGM-coated homogeneous half-plane

Now, consider the Hooke's law in the region $-h<y<0$ the plane contact problem can be in the form

$$\sigma_{1xx}(x,y)=\frac{\mu(y)}{\kappa-1}\left[(\kappa+1)\frac{\partial u_1}{\partial x}+(3-\kappa)\frac{\partial v_1}{\partial y}\right], \tag{4}$$

$$\sigma_{1yy}(x,y)=\frac{\mu(y)}{\kappa-1}\left[(3-\kappa)\frac{\partial u_1}{\partial x}+(\kappa+1)\frac{\partial v_1}{\partial y}\right], \tag{5}$$

$$\sigma_{1xy}(x,y)=\mu_1(y)\left[\frac{\partial u_1}{\partial y}+\frac{\partial v_1}{\partial x}\right], \tag{6}$$

where κ is given by $\kappa=(3-v)/(1+v)$ for the case of generalized plane stress and $\kappa=3-v$ m for the case of plane strain.

For medium 2, $\mu(y)$ is replaced by μ_2 consequently,

$$(\kappa+1)\frac{\partial^2 v_1}{\partial y^2}+(\kappa-1)\frac{\partial^2 v_1}{\partial x^2}+2\frac{\partial^2 u_1}{\partial x\partial y}+\gamma(3-\kappa)\frac{\partial u_1}{\partial x}+\gamma(\kappa+1)\frac{\partial v_1}{\partial y}=0, \qquad -h\prec y\prec 0, \tag{7}$$

$$(\kappa+1)\frac{\partial^2 u_1}{\partial x^2}+(\kappa-1)\frac{\partial^2 u_1}{\partial y^2}+2\frac{\partial^2 v_1}{\partial x\partial y}+\gamma(\kappa-1)\frac{\partial u_1}{\partial y}+\gamma(\kappa-1)\frac{\partial v_1}{\partial x}=0, \qquad -h\prec y\prec 0, \tag{8}$$

$$(\kappa+1)\frac{\partial^2 u_2}{\partial x^2}+(\kappa-1)\frac{\partial^2 u_2}{\partial y^2}+2\frac{\partial^2 v_2}{\partial x\partial y}=0, \qquad -\infty \prec y \prec -h, \tag{9}$$

$$(\kappa+1)\frac{\partial^2 v_2}{\partial y^2}+(\kappa-1)\frac{\partial^2 v_2}{\partial x^2}+2\frac{\partial^2 u_2}{\partial x\partial y}=0, \qquad -\infty \prec y \prec -h. \tag{10}$$

the displacement components, $u_1(x,y)$, $u_2(x,y)$, $v_1(x,y)$, $v_2(x,y)$ can be calculated in detail using Fourier transforms [12].

Now consider a flat rigid stamp with mixed boundary value problems where the tractions σ and τ are equal to zero out of the contact region and in the contact region defined by $a \prec x \prec b$ the displacement components are given. See Fig.(2) where the stamp profile is given by

$$v_1(x,0)=-v_0, \qquad \frac{\partial}{\partial x}v_1(x,0)=0, \tag{11}$$

where v_0 is a given constant and the displacement of the surface is discussed in detail on[12] such that,

$$-\omega\tau(x)+\frac{1}{\pi}\int_{-a}^{b}\left[\frac{1}{t-x}-k_{11}(t,x)\right]\sigma(t)dt-\frac{1}{\pi}\int_{-a}^{b}\tau(t)k_{12}(t,x)dt=f(x), \quad -a \prec x \prec b, \tag{12}$$

$$\omega\sigma(x)+\frac{1}{\pi}\int_{-a}^{b}\left[\frac{1}{t-x}-k_{21}(t,x)\right]\tau(t)dt-\frac{1}{\pi}\int_{-a}^{b}\sigma(t)k_{22}(t,x)dt=g(x), \quad -a \prec x \prec b, \tag{13}$$

where k_{ij} are known kernels and $g(x)$, $f(x)$ are given by the form

$$f(x)=\lambda\frac{\partial}{\partial x}v_1(x,0), \qquad g(x)=\lambda\frac{\partial}{\partial x}u_1(x,0), \qquad \lambda=\frac{4\mu_1}{\kappa+1}, \qquad \omega=\frac{\kappa-1}{k+1}. \tag{14}$$

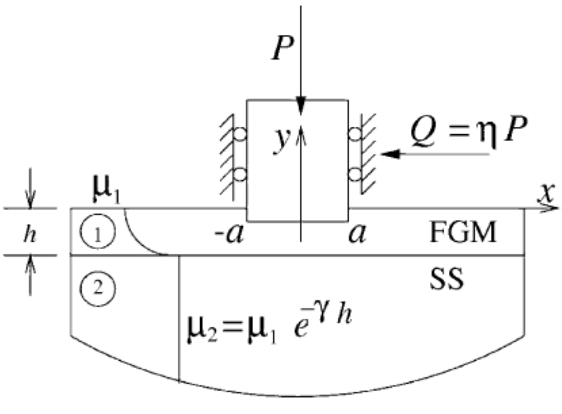

Fig.(2) The geometry of the flat stamp problem.

Assuming that the stamp moves relative to the substrate, the coefficient of friction η in the contact region is constant and the friction is one of the Coulomb types. This yields

$$\sigma_{1yy}(x,0)=\sigma(x)=-P(x), \tag{15}$$

$$\sigma_{1xy}(x,0)=\tau(x)=-\eta P(x), \tag{16}$$

where is $P(x)$ the unknown pressure on the contact region and can be found by solving the following singular integral equation

$$\omega\eta P(x)+\frac{1}{\pi}\int_{-a}^{b}\left[-\frac{1}{t-x}+k_{11}(t,x)+\eta k_{12}(t,x)\right]P(t)dt=f(x), \quad -a\prec x\prec b, \tag{17}$$

that can be rewritten in the form

$$P(x)+\int_{-a}^{b}\tilde{K}(x,t)P(t)dt=\tilde{f}(x), \quad -a\prec x\prec b, \tag{18}$$

where $\tilde{f}(x)=\dfrac{f(x)}{\omega\eta}$, $\tilde{K}(x,t)=\dfrac{\left[\dfrac{1}{t-x}-k_{11}(t,x)-\eta k_{12}(t,x)\right]}{\omega\eta\pi}$.

Which can be calculated by applying the Barycentric Lagrange Interpolation technique [1]. We begin by interpolating the unknown $P(x)$ and data functions $\tilde{f}(x)$ by using the advanced single matrix form barycentric interpolate polynomials; each is expressed through four matrices, and one of which is the monomial basis functions matrix.

$$\tilde{p}_n(x)=X(x)CWP,$$

$$\tilde{f}_n(x)=X(x)CWF=X(x)\phi F \quad ;CW=\phi, \tag{19}$$

The kernel $\tilde{K}(x,t)$ is singular when $x\to t$, which is interpolated twice. the first interpolation is performed with respect to *x*, and the second is performed with respect to *t*. so that we can obtain the double interpolant polynomial of two variables *x* and *t*. Thus, we adopt an approach based on the appropriate choice of two different sets of nodes: the first set is distributed on the right-half interval of the integration domain $\left[-a,\dfrac{b+a}{2}\right]$, and the second set of nodes is distributed on the left-half interval $\left[\dfrac{a+b}{2},b\right]$. This ensures that the difference between the kernel's two variables always remains positive, thus completely erasing the singularity of the kernel and expressing it through five matrices, of which two are monomial basis matrices.

$$\tilde{K}_{(n,n)}(x,t)=\mathrm{X}(x)A\mathrm{K}B\mathrm{X}^T(t)=\mathrm{X}(x)S\mathrm{X}^T(t), \tag{20}$$

let $S=A\mathrm{K}B$. Such that Eq.(18) can be of the form

$$(I-S\mathrm{H})\phi P=\phi F, \quad P=\phi^{-1}\mathrm{M}^{-1}\phi F; \quad \mathrm{M}=(I-S\mathrm{H}). \tag{21}$$

For $\Omega = M^{-1}\phi F = [\gamma_i]_{i=0}^{n}$.

Finally, the interpolant polynomial solution of Eq.(18) can be given by

$$\tilde{P}_n(x) = X(x)\phi\phi^{-1}M^{-1}\phi F = X(x)\Omega. \tag{23}$$

$$\tilde{p}_n(x) = \sum_{i=0}^{n} \gamma x^i. \tag{24}$$

3. Numerical examples

Based on the presented technique, we designed MATLAB R2019b code for the solution of two weakly singular Fredholm integral equations of the second kind resulting from the study of the flat stamp. We find the stress intensity factors and stress distribution on the surface of the FGM coating loaded by a flat stamp and compare the obtained results with the solutions declared in [12].

Example 1

consider a flat stamp with $a/h = 0.1$, $v = 0.3$. Table 1 indicates the powers of stress singularity, β and α, respectively at the leading $x = -a$ and the trailing $x = a$ ends of the stamp at a stiffness ratio $\Gamma_3 = 8, 2, 1, 1/2, 1/8$ corresponding to different values of $\eta = 0.1, 0.3, 0.5$ obtained from the Barycentric Lagrange Interpolation technique and Table 2 represents the powers of stress singularity, β and α, respectively at the leading $x = -a$ and the trailing $x = a$ ends of the stamp aa t stiffness ratio $\Gamma_3 = 8, 2, 1, 1/2, 1/8$ corresponding to different values of $\eta = 0.1, 0.3, 0.5$ obtained from the complex function theory method [12] .

Table 1 Stress intensity factors for flat stamp using Barycentric Lagrange Interpolation technique for Example 1

Γ_3	$\eta=0.1$, $\alpha=-0.5091$, $\beta=-0.4909$		$\eta=0.3$, $\alpha=-0.5272$, $\beta=-0.4728$		$\eta=0.5$, $\alpha=-0.5452$, $\beta=-0.4548$	
	$k_1(-a)/Pa^{\alpha}$	$k_1(a)/Pa^{\beta}$	$k_1(-a)/Pa^{\alpha}$	$k_1(a)/Pa^{\beta}$	$k_1(-a)/Pa^{\alpha}$	$k_1(a)/Pa^{\beta}$
8	0.2765	0.2829	0.2695	0.2879	0.2610	0.2922
2	0.3019	0.3043	0.2987	0.3056	0.2945	0.3059
1	0.3177	0.3178	0.3167	0.3167	0.3148	0.3147
½	0.3362	0.3337	0.3377	0.3301	0.3381	0.3258
1/8	0.3851	0.3765	0.3929	0.3669	0.3995	0.3568

Table 2 Stress intensity factors for flat stamp using the complex function theory method for Example 1

Γ_3	$\eta=0.1$, $\alpha=-0.5091$, $\beta=-0.4909$		$\eta=0.3$, $\alpha=-0.5272$, $\beta=-0.4728$		$\eta=0.5$, $\alpha=-0.5452$, $\beta=-0.4548$	
	$k_1(-a)/Pa^{\alpha}$	$k_1(a)/Pa^{\beta}$	$k_1(-a)/Pa^{\alpha}$	$k_1(a)/Pa^{\beta}$	$k_1(-a)/Pa^{\alpha}$	$k_1(a)/Pa^{\beta}$
8	0.2769	0.2833	0.2696	0.2885	0.2615	0.2926
2	0.3025	0.3048	0.2991	0.3060	0.2949	0.3062
1	0.3182	0.3182	0.3171	0.3171	0.3151	0.3151
½	0.3366	0.3341	0.3382	0.3305	0.3386	0.3261
1/8	0.3855	0.3768	0.3933	0.3673	0.3999	0.3572

Fig.(3) represents the difference between stress distribution $\frac{\sigma_{yy}}{\sigma_0}$ on the surface of the FGM coating loaded by a flat stamp obtained by Barycentric Lagrange Interpolation technique with red color and that obtained from the complex function theory method with blue color, for $\Gamma_3=8$, where $a/h=0.1$, $\eta=0.3$.

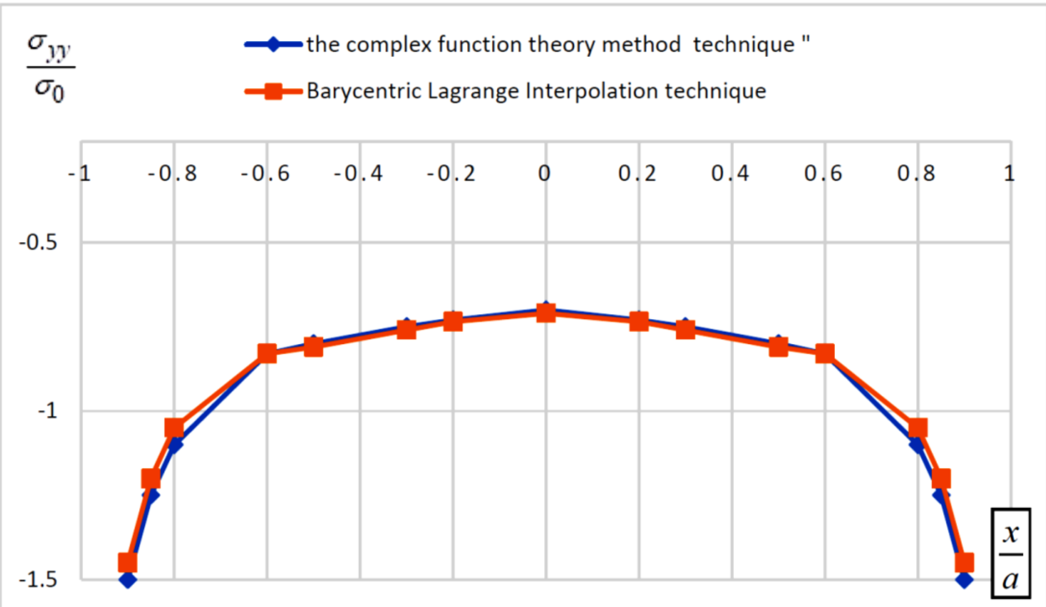

Fig.(3) Stress distribution $\frac{\sigma_{yy}}{\sigma_0}$ on the surface of the FGM coating loaded by a flat stamp

Example 2

Consider a flat stamp with $a/h=0.5$, $v=0.3$. Table 3 indicates the powers of stress singularity, β and α, respectively at the leading $x=-a$ and the trailing $x=a$ ends of the stamp at a stiffness ratio $\Gamma_3=8,\ 2,\ 1,\ 1/2,\ 1/8$ corresponding to different values of $\eta=0.1,\ 0.3,\ 0.5$ obtained from the Barycentric Lagrange Interpolation technique and Table 4 represents the powers of stress singularity, β and α, respectively at the leading $x=-a$ and the trailing $x=a$ ends of the stamp aa t stiffness ratio $\Gamma_3=8,\ 2,\ 1,\ 1/2,\ 1/8$ corresponding to different values of $\eta=0.1,\ 0.3,\ 0.5$ obtained from the complex function theory method [12] .

Fig.(4) represents the difference between stress distribution $\frac{\sigma_{yy}}{\sigma_0}$ on the surface of the FGM coating loaded by a flat stamp obtained by Barycentric Lagrange Interpolation technique with red color and that obtained from the complex function theory method with blue color, for $\Gamma_3=8$, where $a/h=0.5$, $\eta=0.3$.

Table 3 Stress intensity factors for flat stamp using Barycentric Lagrange Interpolation technique for Example 2

Γ_3	$\eta=0.1$ $\alpha=-0.5091$ $\beta=-0.4909$		$\eta=0.3$ $\alpha=-0.5272$ $\beta=-0.4728$		$\eta=0.5$ $\alpha=-0.5452$ $\beta=-0.4548$	
	$k_1(-a)/Pa^{\alpha}$	$k_1(a)/Pa^{\beta}$	$k_1(-a)/Pa^{\alpha}$	$k_1(a)/Pa^{\beta}$	$k_1(-a)/Pa^{\alpha}$	$k_1(a)/Pa^{\beta}$
8	0.1969	0.2764	0.1749	0.2418	0.1545	0.2631
2	0.2653	0.2736	0.2560	0.2809	0.2462	0.2872
1	0.3178	0.3178	0.3167	0.3167	0.3147	0.3148
½	0.3892	0.3798	0.3975	0.3692	0.4049	0.3585
1/8	0.6174	0.5839	0.6506	0.5508	0.6829	0.5181

Table 4 Stress intensity factors for flat stamp using the complex function theory method for Example 2

Γ_3	$\eta=0.1$ $\alpha=-0.5091$ $\beta=-0.4909$		$\eta=0.3$ $\alpha=-0.5272$ $\beta=-0.4728$		$\eta=0.5$ $\alpha=-0.5452$ $\beta=-0.4548$	
	$k_1(-a)/Pa^{\alpha}$	$k_1(a)/Pa^{\beta}$	$k_1(-a)/Pa^{\alpha}$	$k_1(a)/Pa^{\beta}$	$k_1(-a)/Pa^{\alpha}$	$k_1(a)/Pa^{\beta}$
8	0.1973	0.2740	0.1754	0.2422	0.1549	0.2635
2	0.2657	0.2740	0.2565	0.2814	0.2467	0.2876
1	0.3182	0.3182	0.3171	0.3171	0.3151	0.3151
½	0.3895	0.3800	0.3979	0.3696	0.4053	0.3587
1/8	0.6178	0.5844	0.6510	0.5511	0.6834	0.5185

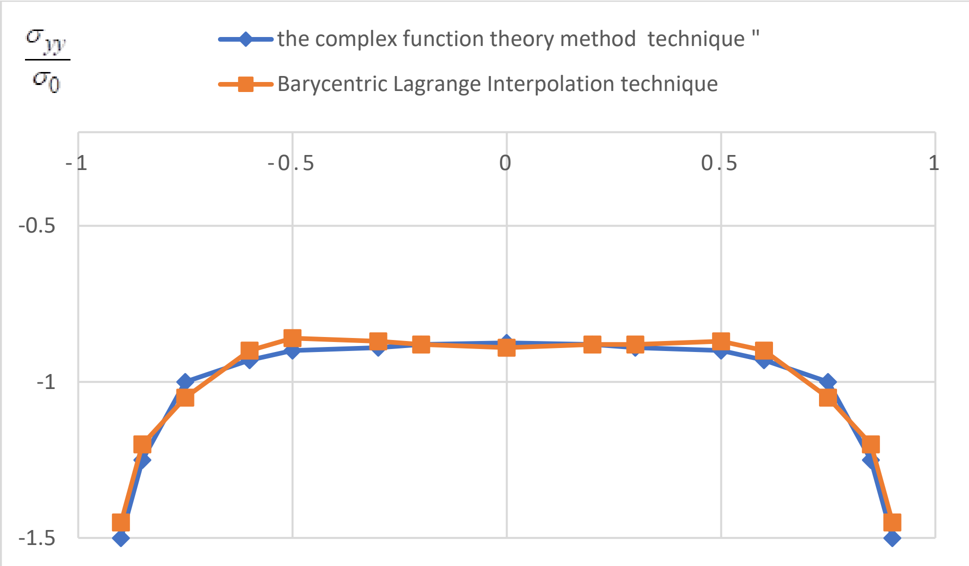

Fig.(4) Stress distribution $\frac{\sigma_{yy}}{\sigma_0}$ on the surface of the FGM coating loaded by a flat stamp

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